

**EELE 477**  
**Digital Signal Processing**

8a

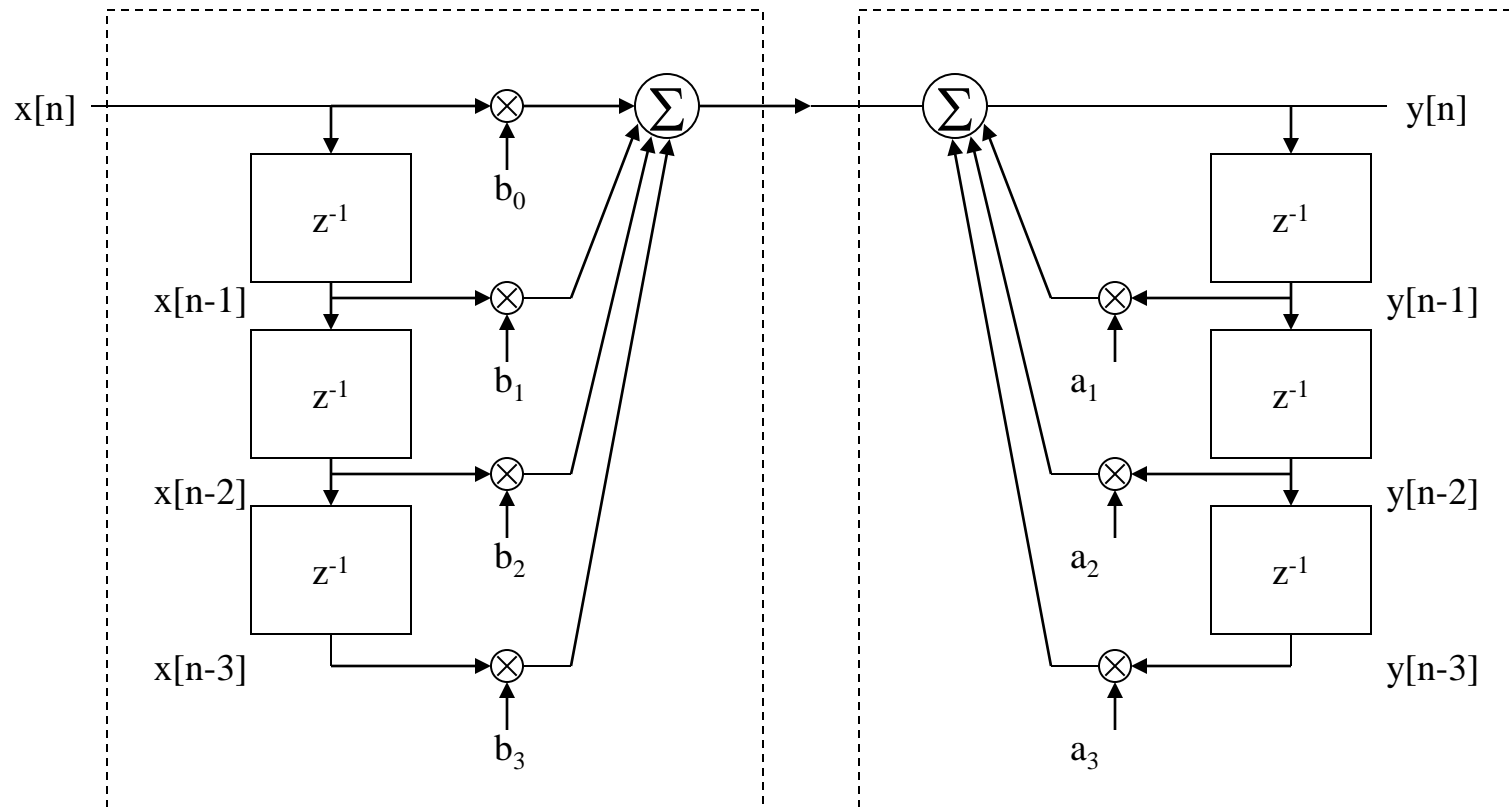
**IIR Systems**

# General Difference Equation

- FIR: output depends on current and past *inputs* only
- IIR: output depends on current and past inputs and *past outputs*

$$y[n] = \sum_{k=0}^M b_k x[n-k] + \sum_{l=1}^N a_l y[n-l]$$

# IIR Block Diagram



# IIR Recursion

- Since current output depends on past outputs, we must either know or assume the prior state
- Generally use *initial rest* assumptions:
  - Input is a right-handed sequence (zero before some  $n=n_0$ )
  - Output is zero prior to  $n_0$

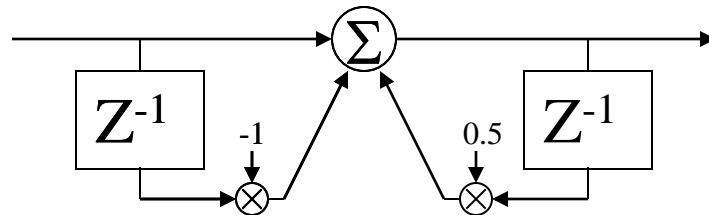
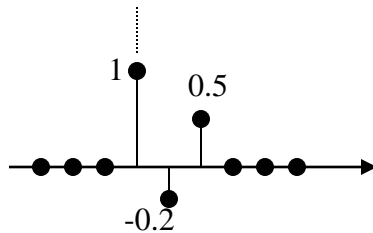
# IIR Recursion (cont.)

- Example: compute response of system described by:

$$y[n] = x[n] - x[n-1] + 0.5y[n-1]$$

with input:

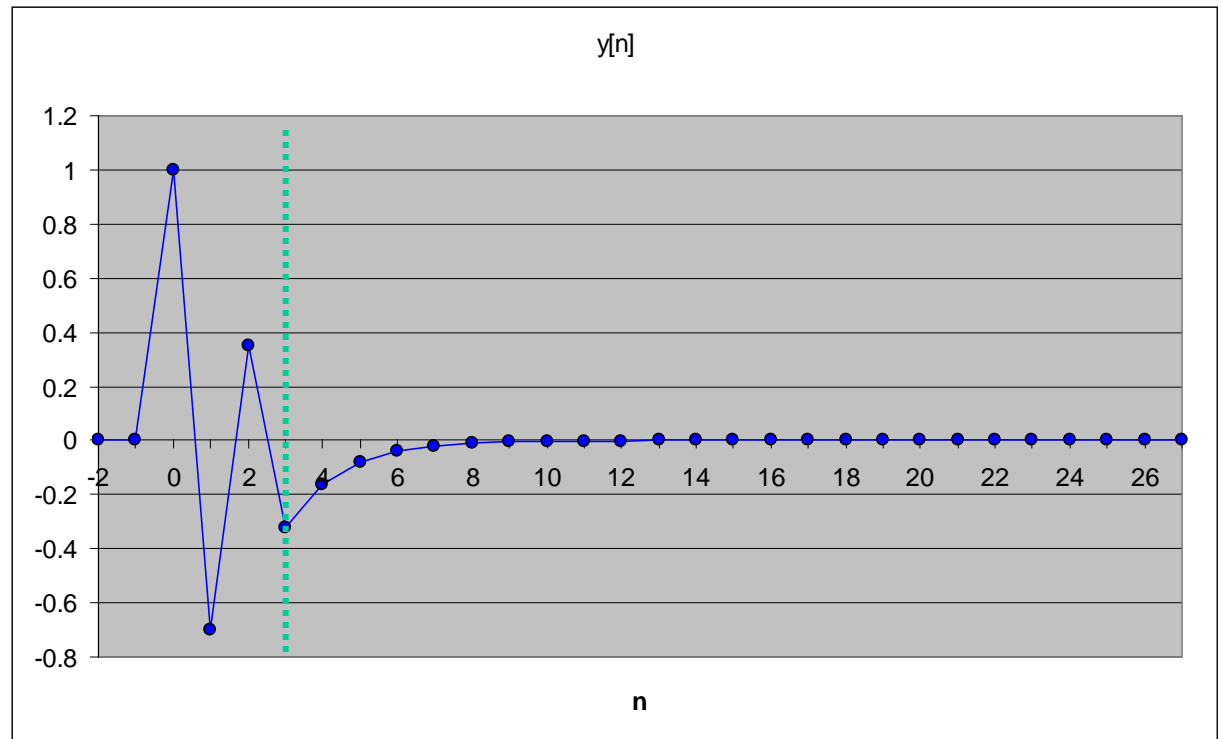
$$x[n] = \delta[n] - 0.2\delta[n-1] + 0.5\delta[n-2]$$



# IIR Recursion (cont.)

- Assume initial rest

n	x[n]	y[n]
-2	0	0
-1	0	0
0	1	1
1	-0.2	-0.7
2	0.5	0.35
3	0	-0.325
4	0	-0.1625
5	0	-0.08125
6	0	-0.04063
7	0	-0.02031
8	0	-0.01016
9	0	-0.00508
10	0	-0.00254
11	0	-0.00127



# Impulse Response

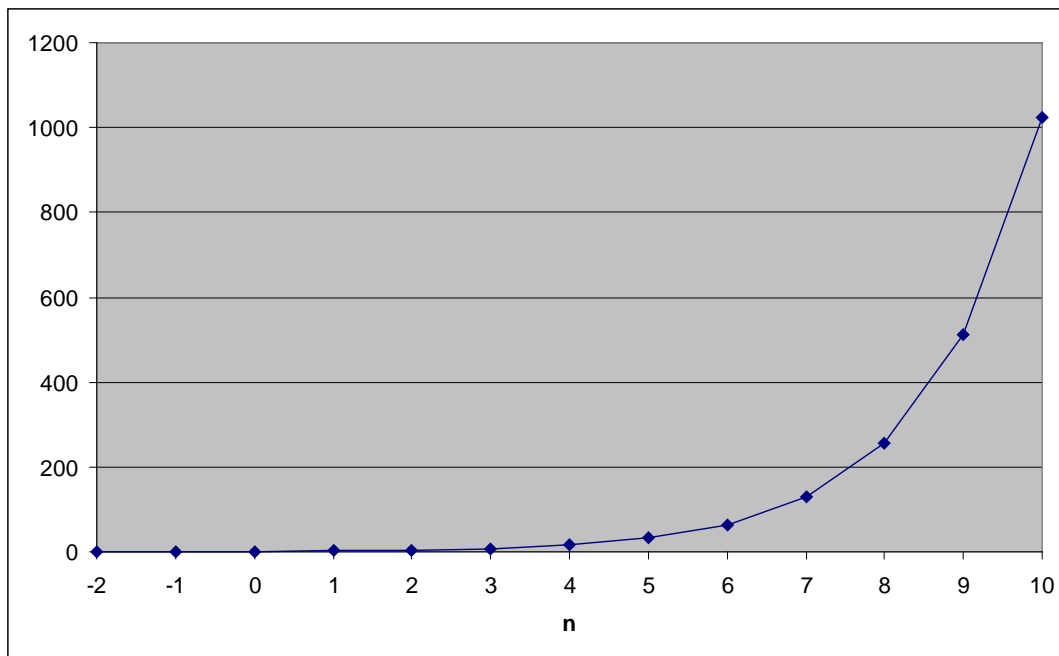
- The general IIR system expression is linear and time invariant (assuming initial rest)
- Output is given by the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Due to recursion, unit sample response can be of infinite extent

# Response Example

- Example:  $y[n] = x[n] + 2y[n - 1]$
- Unbounded unit sample response



$$h[n] = 2^n u[n]$$



# Closed Form Summation

- Some responses can be described in closed form. One useful example:

$$\sum_{k=L_1}^{L_2} r^k = \begin{cases} \frac{r^{L_1} - r^{(L_2+1)}}{1-r}, & r \neq 1 \\ L_2 - L_1 + 1, & r = 1 \end{cases}$$

# Response Behavior

- Consider unit step response of simple IIR system:  $y[n] = a_1 y[n-1] + x[n]$

n	x[n]	y[n]
-1	0	0
0	1	1
1	1	$1+a_1$
2	1	$1+a_1+a_1^2$
3	1	$1+a_1+a_1^2+a_1^3$
4	1	$1+a_1+a_1^2+a_1^3+a_1^4$
5	1	$1+a_1+a_1^2+a_1^3+a_1^4+a_1^5$
...	1	...
n	1	$\sum_{k=0}^n a_1^k$

note that unit sample response is  
 $h[n] = a_1^n u[n]$

in the limit  $n \rightarrow \infty$ ,

$$y[n] = \frac{1 - a_1^{n+1}}{1 - a_1} \rightarrow \frac{1 - a_1^\infty}{1 - a_1}$$

# Response Behavior (cont.)

- Now consider what happens if:

$|a_1| < 1$ : sum converges to  $\frac{1}{1-a_1}$

$|a_1| > 1$ : sum diverges

$|a_1| = 1$ : sum diverges if  $a_1 = 1$ , or alternates  
[1,0,1,0,...] if  $a_1 = -1$

# Response Behavior (cont.)

- In general, we can use convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- In this example with unit step input:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} u[k]a_1^{n-k}u[n-k] \\ &= \begin{cases} 0, & n < 0 \\ \sum_{k=0}^n a_1^{n-k}, & n \geq 0 \end{cases} \\ &= a_1^n \frac{1 - \left(\frac{1}{a_1}\right)^{n+1}}{1 - \left(\frac{1}{a_1}\right)} = \frac{1 - a_1^{n+1}}{1 - a_1} \end{aligned}$$

# IIR System Function

- General first-order IIR:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

- Take z-transform of both sides:

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

- Collect terms

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

# IIR System Function (cont.)

- Since  $Y(z)=H(z)X(z)$ , we can write  $H(z)=Y(z)/X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(b_0 + b_1 z^{-1})}{(1 - a_1 z^{-1})}$$

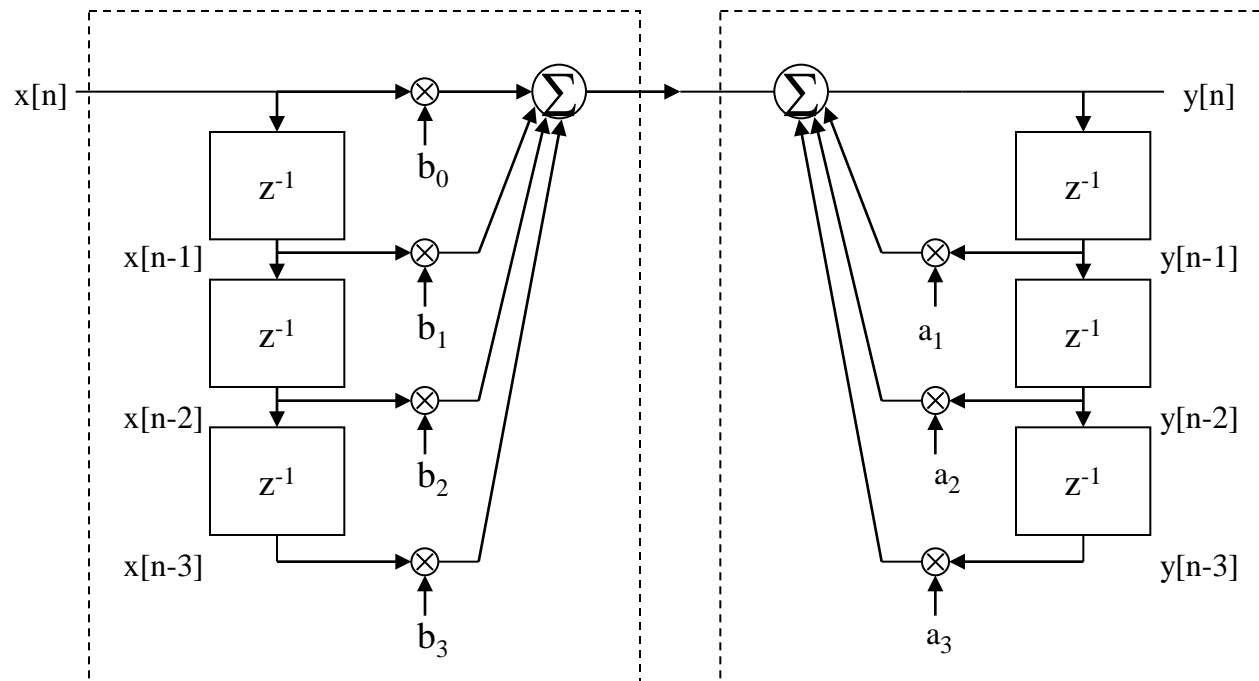
- Note form of numerator and denominator: feed forward on top, 1 with *negative* of feed back on bottom

# Poles and Zeros

- The numerator polynomial has roots that are the *zeros* of the system
- The denominator polynomial has roots that are the *poles* of the system
- If the order of the numerator and denominator polynomials differ, there will be zeros and/or poles at 0 or infinity

# Structures: Direct Form I

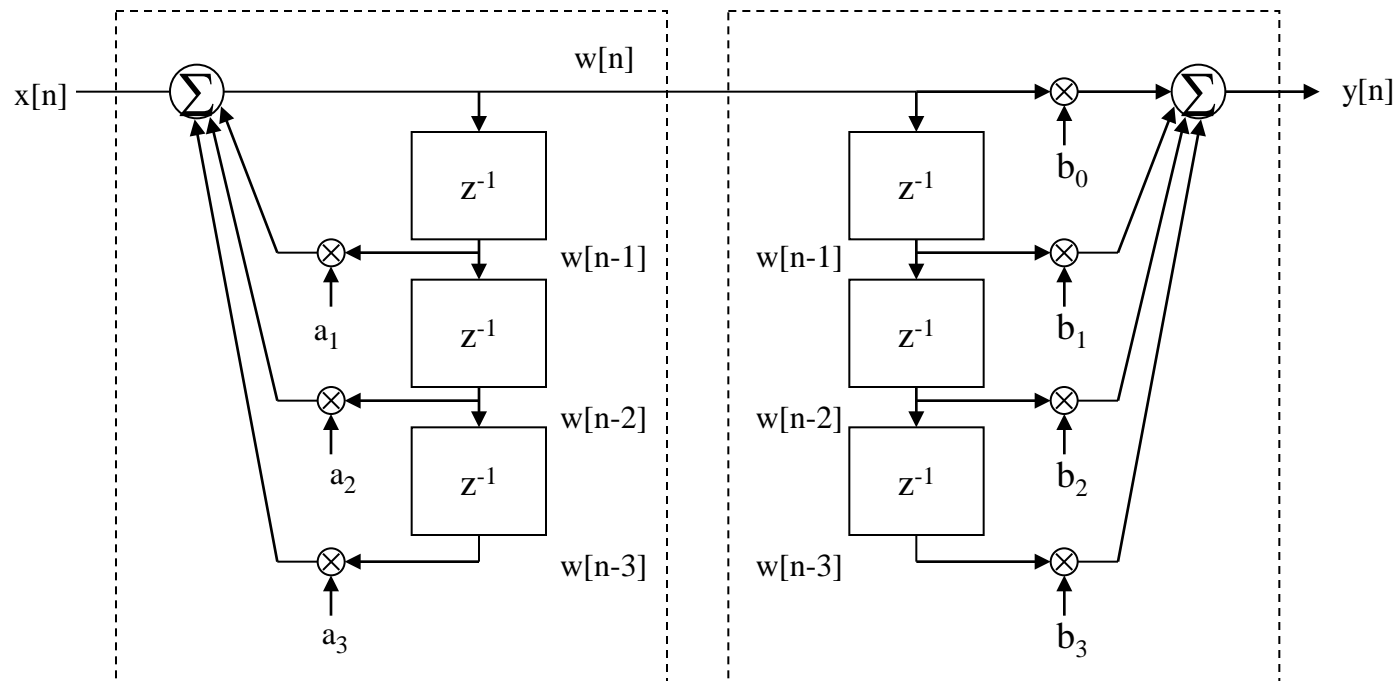
- Direct Form I: feed-forward calculation followed by feedback calculation





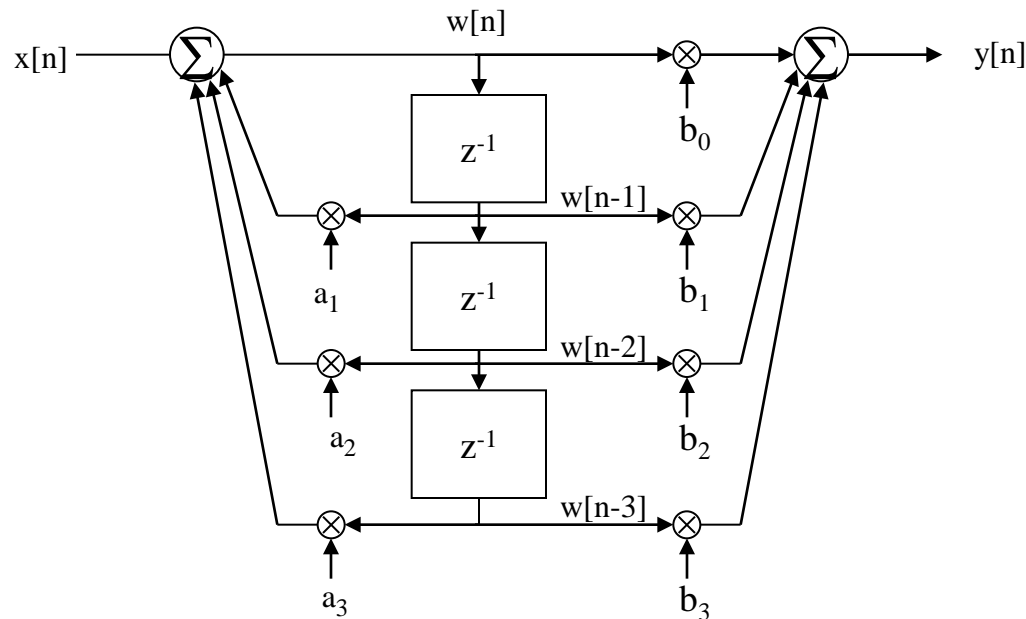
# Direct Form II

- Direct Form II: feedback first, then feed-forward (OK because *LTI* system)



# Direct Form II (cont.)

- Note that delay lines contain the same values, so they can be merged.

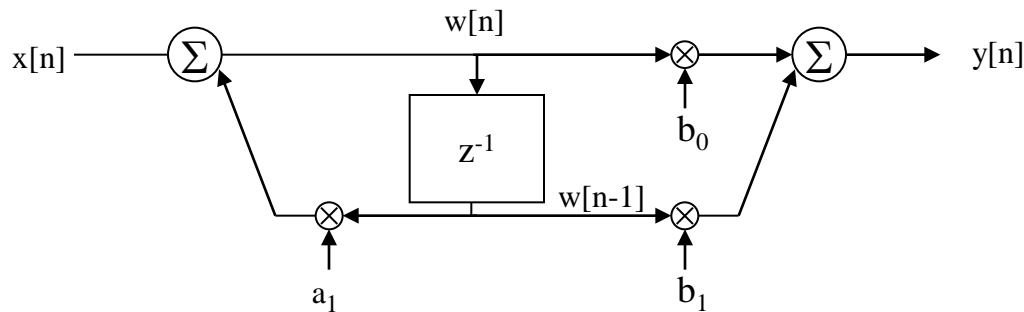


# Transpose Forms

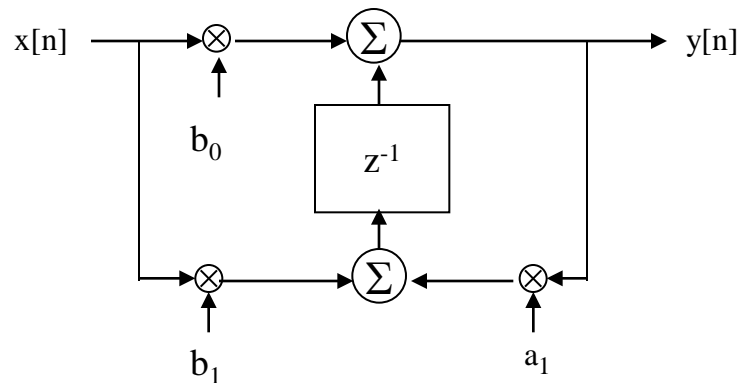
- Oddly enough, the delay/branch/sum block diagram can be “run in reverse” to create an equivalent transpose form:
  - Reverse all arrows in block diagram
  - Branch points become sum points; sum points become branch points
  - Input and Output are exchanged

# Transpose Example

- Original: first-order direct form II



- Transposed version



# Region of Convergence

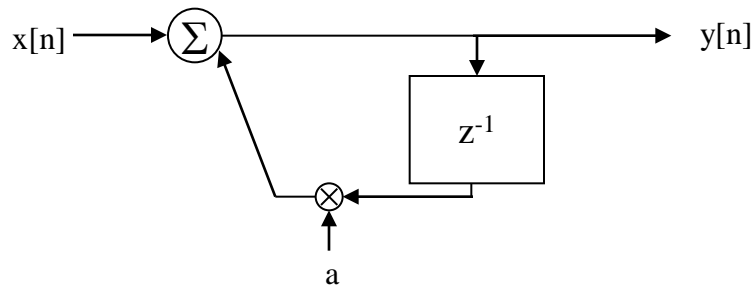
- The z-transform of the unit sample response is the system function,  $H(z)$
- The z-transform is computed from

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- This was OK for FIR, but for IIR we must be sure that the infinite sum actually converges!

# Region of Convergence (cont.)

- Example:  $h[n]=a^n u[n]$

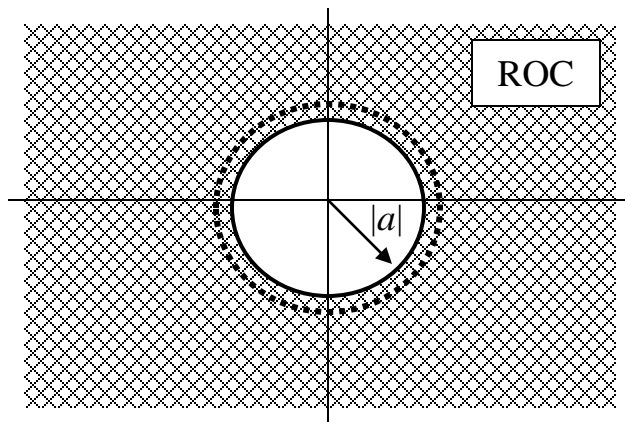


$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- If  $|az^{-1}| < 1$ , then sum converges  $\sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$

# Region of Convergence (cont.)

- Since  $|az^{-1}| < 1$ , this implies  $|z| > |a|$ .
- This means that the ROC will be all values of  $|z|$  outside a radius  $r = |a|$



# Region of Convergence (cont.)

- In order for a causal system to be stable, the ROC must include the unit circle.
- The poles of the system define the edge of the ROC.
- This means: a stable, causal system will have all its poles *inside* the unit circle, so that the ROC extends outward (including the unit circle) from the largest pole.