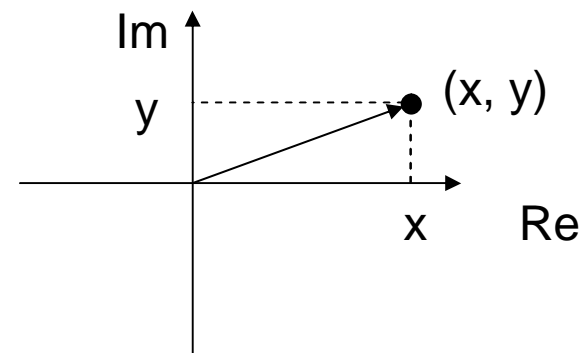


EE 477
Digital Signal Processing

2
Complex Exponentials

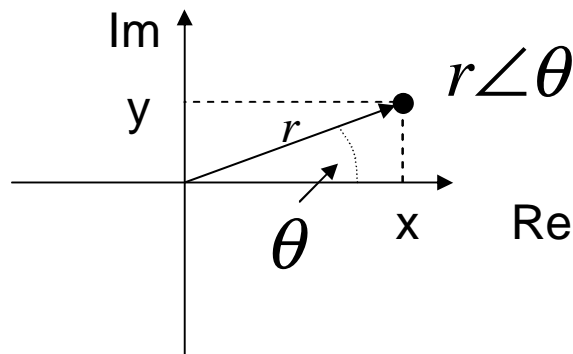
Complex Numbers

- Represent a number in terms of a *real* part and an *imaginary* part.
- The imaginary part simply means that it contains a $\sqrt{-1}$ factor.
- Example: say z is a complex number. Then
$$z = (x, y) = x + jy = \text{Re}\{ z \} + j\text{Im}\{ z \}$$
- Rectangular form:



Polar Form

- Often convenient to express complex number as a *vector* in the complex plane: polar form



$$r = \sqrt{x^2 + y^2}$$

Polar and Rectangular Relationships

- $x = r \cos(\theta)$ $y = r \sin(\theta)$
- $z = r \cos(\theta) + j r \sin(\theta)$

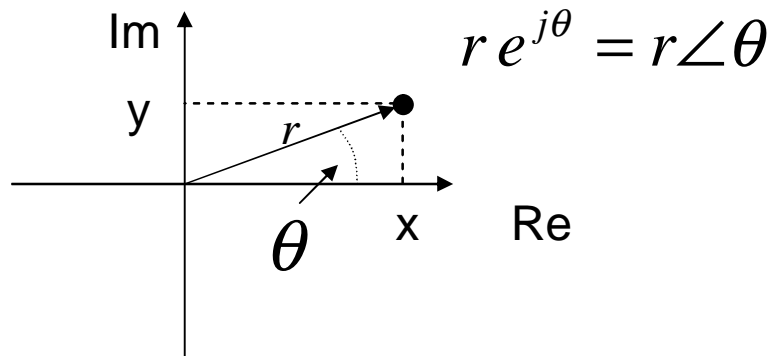
$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

- Note that $\arctan()$ must be unambiguous (clear about which quadrant)

Euler's Formula

- An interesting insight:

$$e^{j\theta} = \cos(\theta) + j \sin(\theta) = 1 \angle \theta$$



$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Exponential Form

- Complex exponential (polar) form is appropriate when *multiplying* or *dividing* complex numbers. Exponents add or subtract conveniently:

$$r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$r_1 e^{j\theta_1} \div r_2 e^{j\theta_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

Complex Rectangular Form

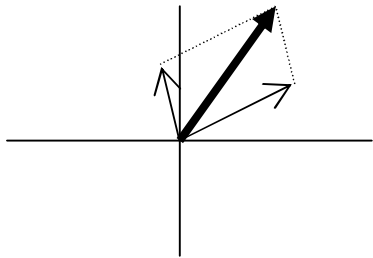
- Rectangular (Cartesian) form is most appropriate when adding or subtracting complex numbers. Real and imaginary parts are treated separately:

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2)$$

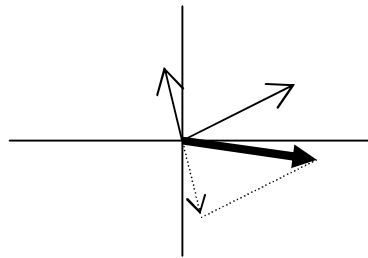
Geometric Viewpoint

- Addition: construct vector head-to-tail sequence
- Subtraction: find $-z_1$, then head-to-tail
- Multiplication: multiply magnitudes, add angles (rotation)
- Division: divide magnitudes, subtract angles
- Inverse: invert magnitude, negate the angle
- Conjugate: flip vector across horizontal (real) axis

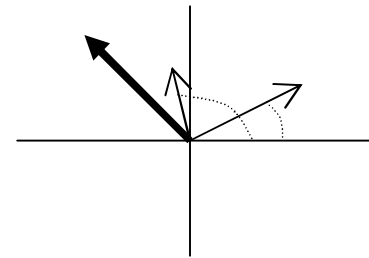
Geometric Viewpoint (cont.)



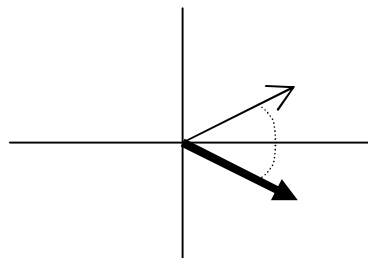
Sum



Difference



Product



Conjugate

Complex Exponential Signals

- Now consider allowing angle to be a function of time:

$$\tilde{x}(t) = Ae^{j(\omega_0 t + \phi)} = A \cos(\omega_0 t + \phi) + jA \sin(\omega_0 t + \phi)$$

- NOTE that we can get the real signal $x(t)$ simply by taking the real part of $\tilde{x}(t)$:

$$x(t) = \text{Re}\{Ae^{j(\omega_0 t + \phi)}\} = A \cos(\omega_0 t + \phi)$$

Phasor Concept

- Pull out the complex amplitude:

$$\tilde{x}(t) = Ae^{j(\omega_0 t + \phi)} = Ae^{j\phi} e^{j\omega_0 t} = \tilde{X}e^{j\omega_0 t}$$

- This \tilde{X} is called a *phasor*. Combining with the time variation term, this is a *rotating phasor*.
- The phase shift defines where the rotating vector is pointing at $t=0$.

Phasor Addition

- Often need to add several sinusoids with the same frequency but different amplitude and phase:

$$x(t) = \sum_{k=1}^N A_k \cos(\omega_0 t + \phi_k) = \text{Re} \left\{ \sum_{k=1}^N \tilde{X}_k e^{j\omega_0 t} \right\}$$

- NOTE that phasor factors can be summed! Simpler than trig identities.
- Convert to rectangular, sum real, sum imag, convert back to polar form.